Sketch of a graph

1. Let $f(x) = x^3 - 6x^2 + 9x + 1$.
   (1) Find the intervals where f is increasing and the intervals where f is decreasing.
   (2) Find the local maximum point and the local minimum point of f.
   (3) Find the intervals where f is concave up and the intervals where f is concave down.
   (4) Find the inflection point of f.
   (5) Sketch f.

Sol (1) $f'(x) = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$

$$f'(x) \begin{cases} > 0, & 3 < x \\ < 0, & 1 < x < 3 \\ > 0, & x < 1 \end{cases}$$

$\therefore$ f is increasing on $(-\infty, 1]$,
decreasing on $[1, 3]$,
increasing on $[3, \infty)$

(2) $f(1) = 5$

$(1, f(1)) = (1, 5)$ is a local max. point of f.

$f(3) = 1$

$(3, f(3)) = (3, 1)$ is a local min. point of f.

(3) $f''(x) = 6x - 12 = 6(x-2)$

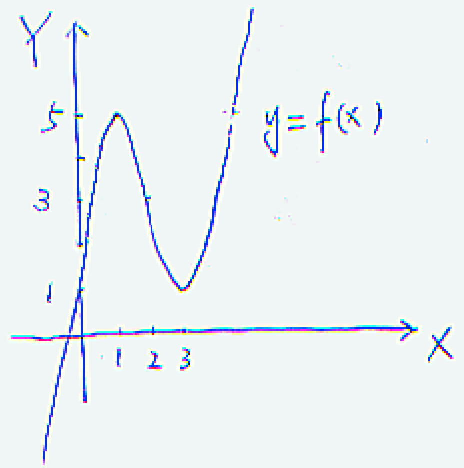
$$f''(x) \begin{cases} > 0, & 2 < x \\ < 0, & x < 2 \end{cases}$$

f is concave up on $[2, \infty)$, and concave down on $(-\infty, 2)$

(4) $f(2)=3$

$(2, f(2))=(2,3)$ is the inflection point of f.

(5)



2. Let $f(x)=\dfrac{x^2-3}{x^3}$.
   (1) Find the asymptote of f.
   (2) Find the x-intercepts and y-intercepts of f.
   (3) Show that f is an odd function.
   (4) Find the intervals where f is increasing and the intervals where f is decreasing.
   (5) Find the local maximum point and the local minimum point of f.
   (6) Find the intervals where f is concave up and the intervals where f is concave down.
   (7) Find the inflection points of f.
   (8) Sketch f.

Sol (1) $x^3=0 \Rightarrow x=0$

$\therefore x=0$ is a (vertical) asymptote of f.

$\lim\limits_{x\to\infty} \dfrac{x^2-3}{x^3}=0$

$\therefore y=0$ is a (horizontal) asymptote of f.

(2) $\dfrac{x^2-3}{x^3}=0 \Rightarrow x^2-3=0 \Rightarrow x=\pm\sqrt{3}$

$x=-\sqrt{3}$, $x=\sqrt{3}$ are the x-intercepts of f

No y-intercept.

(3) $f(-x)=\dfrac{(-x)^2-3}{(-x)^3}=-\dfrac{x^2-3}{x^3}=-f(x)$

$\therefore f$ is an odd function.

(4) $f(x) = x^{-1} - 3x^{-3} \Rightarrow f$ is not defined at 0.

$f'(x) = -x^{-2} + 9x^{-4}$

$= -x^{-4}(x^2 - 9)$

$= -x^{-4}(x+3)(x-3)$

f': $- \; -$ | -3 | $+ \; +$ | 0 | $+ \; +$ | 3 | $- \; -$

$\therefore f$ is decreasing on $(-\infty, -3)$,

increasing on $(-3, 0)$

increasing on $(0, 3)$

decreasing on $(3, \infty)$

(5) $(-3, f(-3))$ is the local min. point of f

where $f(-3) = -\frac{2}{9}$

$(3, f(3))$ is the local max. point of f

where $f(3) = \frac{2}{9}$.

(6) $f''(x) = 2x^{-3} - 36x^{-5}$

$= 2x^{-5}(x^2 - 18)$

$= 2x^{-5}(x + 3\sqrt{2})(x - 3\sqrt{2}) = 2(x+3\sqrt{2})\,x^{-5}(x-3\sqrt{2})$

f'': $- \; -$ | $-3\sqrt{2}$ | $+ \; +$ | 0 | $- \; -$ | $3\sqrt{2}$ | $+ \; +$

f is concave down on $(-\infty, -3\sqrt{2})$,

concave up on $(-3\sqrt{2}, 0)$,

concave down on $(0, 3\sqrt{2})$,

concave up on $(3\sqrt{2}, \infty)$

(7) $f(-3\sqrt{2}) = -\frac{5\sqrt{2}}{36}$

$\left(-3\sqrt{2}, -\frac{5\sqrt{2}}{36}\right)$ is an inflection point of f.

$f(3\sqrt{2}) = \frac{5\sqrt{2}}{36}$

$\left(3\sqrt{2}, \frac{5\sqrt{2}}{36}\right)$ is another inflection point of f.

(8)